Worcester County Mathematics League

Varsity Meet 4
March 7, 2012

Coaches' Copy
Rounds, Answers and Solutions

Worcester County Mathematics League

Varsity Meet 4 - March 7, 2012

Round 1: Number Theory

1

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

Note: a subscript indicates a number's base

1. Determine the base B for which the following subtraction is true:

$$\begin{array}{r} 3\;6\;1\;1\;6_B \\ -\quad 3\;4\;7\;1_B \\ \hline 3\;2\;8\;6\;5_B \end{array}$$

2. Find the least common multiple of 11_3, 20_3, and 100_3. Write your answer in base three.

3. The integer N has 12 factors (including 1 and N itself). Determine the maximum number of factors that N^2 could have.

ANSWERS

(1 pt.) 1.______________________

(2 pts.) 2. ______________________

(3 pts.) 3.______________________

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Round 2: Algebra 1 – Open

2

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

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1. Solve for x: $\dfrac{x}{x+1} - \dfrac{1}{x} = 1$

2. Luis and Juan live in cities that are 350 kilometers apart. Assuming that they both drive at constant rates and leave at the same time, if they were to drive toward each other, they would meet in 2 hours. However, if Luis were to drive toward Juan's city and Juan were to drive in the direction opposite to Luis's city, it would take Luis 10 hours to overtake Juan. At what rate of speed does Juan drive (in kilometers per hour)?

3. Let x and y represent consecutive odd integers such that $x < y$. If $3x^2 - 2y = 129$, compute the value of $x + y$.

ANSWERS

(1 pt.) 1. $x =$ ____________________

(2 pts.) 2. ____________________ km / hr

(3 pts.) 3. ____________________

South, Tantasqua, Tahanto

Worcester County Mathematics League

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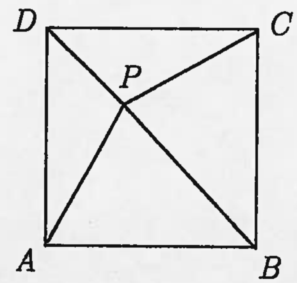
Round 3: Geometry - Open

3

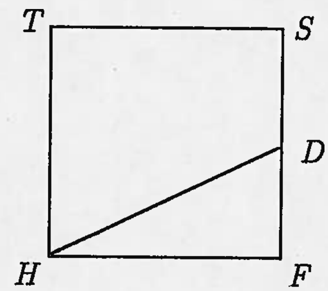
All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED The diagrams are NOT drawn to scale

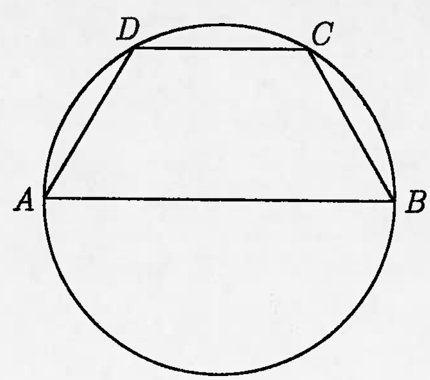
1. The diagram to the right shows square $ABCD$ and point P inside of the square. If $AB = 12$ and the area of ΔAPD is 30, compute the area of ΔBCP.



2. Dustin Pedroia (point D) leaves base F running in a straight line towards base S at k feet per second as suggested by the diagram to the right. After x seconds the area of ΔHFD is $k\%$ of the area of square $HFST$. If the length of FS is 90 feet, compute the value of x.



3. The diagram below shows isosceles trapezoid $ABCD$ (with bases $\overline{AB}$ and $\overline{DC}$) inscribed in a circle such that $\overline{AB}$ is a diameter of the circle and $AD = DC = CB$. If the length of $\overline{AB}$ is 12, compute the exact area of $ABCD$ in simplest radical form.



ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________ seconds

(3 pts.) 3. ____________________

AMSA Charter, Auburn, Hudson

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Round 4: Logarithms, Exponents and Radicals

4

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

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1. Simplify: $4(2)^{-2} - 5 \cdot 2^0 + 4\left(\frac{4}{9}\right)^{-\frac{1}{2}}$

2. Solve for x: $\sqrt{\sqrt[3]{\sqrt{3}+\sqrt{3}+\sqrt{3}}} = 3^{3x}$

3. Find all of the values of y that satisfy the equation $y^{\log_3 y} = y \cdot 3^6$.

ANSWERS

(1 pt.) 1.____________________

(2 pts.) 2. $x=$____________________

(3 pts.) 3.____________________

Oxford, Quaboag, Worcester Academy

Worcester County Mathematics League

Varsity Meet 4 - March 7, 2012
Round 5: Trigonometry - Open

5

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

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1. Find all of the values of x, $0° \le x < 360°$, that satisfy the equation

$$\frac{2}{\sin x} + \frac{\sqrt{3}}{\sin^2 x} = 0.$$

2. If $\sin\left(\frac{\theta}{2}\right) = \frac{1}{4}$, compute the value of $\cos(2\theta)$.

3. Simplify the following expression by writing it in terms of only the tangent function, $\tan x$.

$$\frac{\sin 4x}{1 + 2\cos 2x + \cos 4x} + \frac{3\sin x}{\cos x}$$

ANSWERS

(1 pt.) 1.____________________ degrees

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Quaboag, Hudson, Leicester

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TEAM ROUND

All answers must *either* be in simplest exact form *or* as decimals rounded correctly to at least three decimal places, unless stated otherwise (2 pts. each)

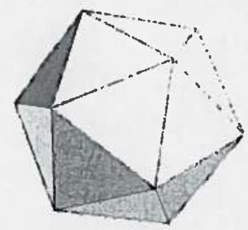
APPROVED CALCULATORS ALLOWED The diagrams are NOT drawn to scale

1. If $2A+3B=5C$ and $3A+2B=4C$, then $18B+6C=kA$ for some integer k. Find the value of k.

2. In a rectangle whose perimeter is 20, the length of the diagonal is d. Compute the sum of all of the possible integer values for d.

3. Find the two smallest positive integers that will have remainders of 1, 2, and 2 upon being divided by 3, 5, and 7 respectively.

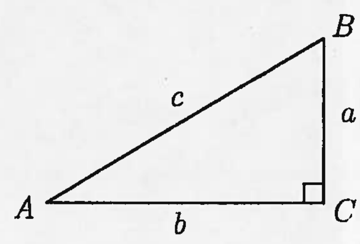
4. The icosahedron illustrated to the right is a solid with 20 faces, each of which is an equilateral triangle. The faces are joined along edges and meet at vertices. For this solid, find the sum of the number of vertices and the number of edges.



5. The three-digit number N = XYZ consists of the consecutive positive integers X, Y and Z such that X >Y > Z. If N is equal to 48 times the sum of the digits of N, determine N.

6. The quantity Y varies directly as the product of a and b and varies inversely as the square of c. If a is increased by 10%, b is decreased by 4% and c is decreased by 5%, then Y changes by X%. Find the value of X to the **nearest whole number** and tell whether the change in Y is an increase or decrease.

7. Determine the largest integer value of $N \le 2012$ for which the expression $\sqrt{N+\sqrt{N+\sqrt{N+\ldots}}}$ converges to a positive integer.



8. Let ΔABC be a right triangle with $\angle C = 90°$, hypotenuse c, and legs a and b as shown to the right. If $a^2-b^2=2ab$, then $\tan^2 A+\tan^2 B=N$, where N is an integer. Compute the value of N.

9. How many zeros are at the end of the expansion of $(5^5)!$, where $n!$ represents n factorial?

Worcester Academy (1 & 6), Algonquin (2 & 9), Quaboag, QSC, Doherty, Auburn, St. John's

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ANSWER SHEET – TEAM ROUND

All answers must *either* be in simplest exact form *or* as decimals rounded correctly to at least three decimal places unless stated otherwise (2 pts. each)

1.______________________________

2.______________________________

3.______________________________

4. ______________________________

5. ______________________________

6. X = _______% and the change is a(n) ____________________

7. ______________________________

8.______________________________

9.______________________________ zeros

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ANSWERS

Round 1

1. 12
2. 1100_3 or 1100
3. 45

Round 2

1. $-\frac{1}{2} = -0.5$
2. 70
3. 16

Round 3

1. 42
2. $\frac{9}{5} = 1\frac{4}{5} = 1.8$
3. $27\sqrt{3}$ (only)

Round 4

1. 2
2. $\frac{1}{12} = 0.08\overline{3}$
3. $27, \frac{1}{9} = 0.\overline{1}$ (need both, either order)

Round 5

1. $240°, 300°$ or 240, 300 (need both, either order)
2. $\frac{17}{32} = 0.53125$
3. $4\tan x$

Team Round

1. 78
2. 17
3. 37, 142 (need both in either order)
4. 42
5. 432
6. 17 and increase
7. 1980
8. 6
9. 781

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BRIEF SOLUTIONS

Round 1

1. In the "B" place notice that B must be borrowed from the "B^2" place giving, in the "B" place, $B+1-7=6 \Rightarrow B=12$.

2. In base 10, $11_3 = 4 = 2^2$, $20_3 = 6 = 2\cdot 3$, and $100_3 = 9 = 3^2$. The least common multiple of 4, 6, and 9 is $2^2\cdot 3^2 = 36$. Converting back to base-3, $36 = 3^3 + 3^2 = 1100_3$.

3. If the prime factorization of n is $n = p_1^k \cdot p_2^l \cdot p_3^m$, for three prime numbers p_1, p_2, p_3, then n has $(k+1)(l+1)(m+1)$ factors. Notice that $12 = 2^2 \cdot 3$, so that the prime factorization of n is in one of the following forms: $p_1 \cdot p_2 \cdot p_3^2$, $p_1^2 \cdot p_2^3$, $p_1 \cdot p_2^5$, or p_1^{11}. As a result the prime factorization of n^2 has one of the following forms: $p_1^2 \cdot p_2^2 \cdot p_3^4$, $p_1^4 \cdot p_2^6$, $p_1^2 \cdot p_2^{10}$, or p_1^{22}. The first of these factorizations has the largest number of factors, namely $(2+1)(2+1)(4+1) = 3\cdot 3\cdot 5 = 45$.

Round 2

1. $\dfrac{x}{x+1} - \dfrac{1}{x} = \dfrac{x^2-x-1}{(x+1)x}$ so that $\dfrac{x^2-x-1}{(x+1)x} = 1 \Rightarrow x^2 - x - 1 = x^2 + x \Rightarrow 2x = -1 \Rightarrow x = -\dfrac{1}{2}$.

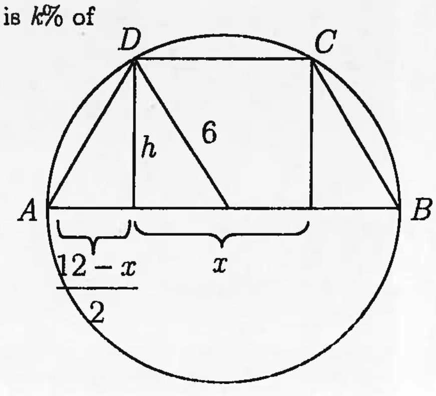
2. Let J and L = the speeds at which Juan and Luis travel at, respectively. The second sentence of the problem implies that $2L + 2J = 350$. The third sentence of the problem implies that $10L - 10J = 350$. Solving these equations simultaneously yields $J = 70$ km/hr (and $L = 135$ km/hr).

3. If x and y are consecutive odd integers, then $y = x+2$ so that $3x^2 - 2y = 129 \Rightarrow 3x^2 - 2x - 4 = 129$ $\Rightarrow 3x^2 - 2x - 133 = 0 \Rightarrow (3x+19)(x-7) = 0 \Rightarrow x = 7$ and $y = 9 \Rightarrow x + y = 16$.

Round 3

1. Note that $BC = AD = 12$. Draw altitudes PE (of ΔAPD) and PF (of ΔBCP) ending on sides AD and BC respectively. Note that E, P, and F are collinear because both PE and PF are parallel to AB. Then, $EP + PF = AB = 12$ or $PF = 12 - EP$. Since the area of ΔAPD = 30, we have $\frac{1}{2}\cdot AD \cdot PE = 6\cdot PE = 30$ $\Rightarrow PE = 5$ and $PF = 7$. Thus, the area of ΔBCP is $\frac{1}{2}\cdot PF \cdot BC = \frac{1}{2}\cdot 7\cdot 12 = 42$.

2. After x seconds, $DF = x\cdot k$ so that the area of $\Delta HFD = \frac{1}{2}\cdot 90\cdot xk$. Since this area is k% of the area of square HFST, we have $\frac{1}{2}\cdot 90\cdot xk = \frac{k}{100}\cdot 90^2 \Rightarrow x = \frac{180}{100} = 1.8$.



3. Let h = the altitude of the trapezoid and let $x = AD = CD = BC$. By the Pythagorean theorem (twice) we have $h^2 = x^2 + \left(\frac{12-x}{2}\right)^2$ and $h^2 + \left(\frac{x}{2}\right)^2 = 36$.

Substitute for h^2 and simplify: $x^2+6x-72=0 \Rightarrow (x-6)(x+12)=0 \Rightarrow x=6$. As a result, $h^2=27 \Rightarrow h=3\sqrt{3}$ and the area of the trapezoid is $\frac{1}{2}\cdot 3\sqrt{3}\cdot\left(6+12\right)=27\sqrt{3}$.

Round 4

1. $4\left(2\right)^{-2}-5\cdot 2^0+4\left(\frac{4}{9}\right)^{-\frac{1}{2}}=\frac{4}{4}-5+4\cdot\frac{3}{2}=1-5+6=2$.

2. $\sqrt{\sqrt[3]{\sqrt{3}+\sqrt{3}+\sqrt{3}}}=3^{3x} \Rightarrow \sqrt{\sqrt[3]{3\sqrt{3}}}=3^{3x}$. Square and then cube both sides to get: $3\sqrt{3}=3^{18x} \Rightarrow 3^{\frac{3}{2}}=3^{18x}$
$\Rightarrow 18x=\frac{3}{2} \Rightarrow x=\frac{3}{36}=\frac{1}{12}$.

3. Take $\log_3$ of both sides to get $\log_3 x^{\log_3 x}=\log_3 3^6 x$ or $\log_3 x\cdot\log_3 x=6\log_3 3+\log_3 x$. This equation is quadratic in $\log_3 x$. Thus, we have $(\log_3 x)^2-\log_3 x-6=0 \Rightarrow (\log_3 x-3)(\log_3 x+2)=0 \Rightarrow \log_3 x=3 \Rightarrow x=27$ or $\log_3 x=-2 \Rightarrow x=\frac{1}{9}$.

Round 5

1. $\frac{2}{\sin x}+\frac{\sqrt{3}}{\sin^2 x}=0 \Rightarrow \frac{2\sin x+\sqrt{3}}{\sin^2 x}=0 \Rightarrow \sin x=-\frac{\sqrt{3}}{2} \Rightarrow x=240°$ and $x=300°$.

2. There several useful versions of the double angle for $\cos 2\theta$, namely $\cos 2\theta=1-2\sin^2\theta$ and $\cos\theta=2\cos^2\theta-1$. In this case, the double angle formula for $\cos 2\left(\frac{\theta}{2}\right)$ gives

$\cos\theta=1-2\sin^2\left(\frac{\theta}{2}\right)=1-2\cdot\left(\frac{1}{4}\right)^2=\frac{7}{8}$. Now, applying the double angle formula again gives

$\cos 2\theta=2\cos^2\theta-1=2\left(\frac{7}{8}\right)^2-1=\frac{17}{32}$.

3. Several applications of the double angle formulas for sine and cosine produce
$$\frac{\sin 4x}{1+2\cos 2x+\cos 4x}=\frac{2\sin 2x\cos 2x}{1+2\cos 2x+2\cos^2 2x-1}=\frac{2\sin 2x\cos 2x}{2\cos 2x(1+\cos 2x)}=\frac{\sin 2x}{1+\cos 2x}=\frac{2\sin x\cos x}{1+2\cos^2 x-1}=\frac{\sin x}{\cos x}=\tan x$$
so that $\frac{\sin 4x}{1+2\cos 2x+\cos 4x}+\frac{3\sin x}{\cos x}=\tan x+3\tan x=4\tan x$.

Team Round

1. $2A+3B=5C$ is equivalent to 1) $4A+6B=10C$ and 2) $8A+12B=20C$. Also, $3A+2B=4C$ is equivalent to 3) $9A+6B=12C$ and 4) $15A+10B=20C$. Subtract equations 1) and 3) to get $-5A=-2C \Rightarrow C=\frac{5}{2}A$.

Then, subtract 2) and 4) to get $-7A+2B=0 \Rightarrow B=\frac{7}{2}A$. Therefore, $18B+6C=18\left(\frac{7}{2}A\right)+6\left(\frac{5}{2}A\right)$

$=63A+15A=78A \Rightarrow k=78$.

2. If we allow the "shape" of the rectangle to vary we would observe two extremes. First, the rectangle could be a square with sides of length 5, and as a result the length of the diagonal would be $5\sqrt{2}$. On the other hand, as one dimension of the rectangle shrinks, the other dimension must increase in order to maintain the condition that the perimeter is 20. The second extreme is a rectangle with one dimension very close to 10 and the other dimension very close to zero. The diagonal of this rectangle would have a length very close to 10. Therefore, the set of all possible diagonal lengths d is $5\sqrt{2} \le d < 10$. Since $5\sqrt{2} \approx 7.1$, the only possible integer lengths for the diagonal are 8 and 9 $\Rightarrow$ their sum is 17.

3. Let $x =$ such a positive integer. Then $x = 3a+1$, $x = 5b+2$, and $x = 7c+2$ for some positive integers a, b, and c. This means that $5b+2 = 7c+2 \Rightarrow b = \frac{7c}{5} \Rightarrow$ we can let $c =$ a multiple of 5. If $c = 5$, then $x = 37$ and $37 \div 3$ leaves a remainder of 1 $\Rightarrow x = 37$ is the smallest such x. Next, If $c = 10$, then $x = 72$ and $72 \div 3$ leaves a remainder of 0. If $c = 15$, then $x = 107$ and $107 \div 3$ leaves a remainder of 2. Finally, if $c = 20$, then $x = 142$ and $142 \div 3$ leaves a remainder of 1 $\Rightarrow x = 142$ is the next smallest such x.

4. The icosahedron has 20 faces and each triangular face has 3 vertices. However, five faces meet at each vertex. Therefore, there are $\frac{1}{5}(3 \times 20) = 12$ vertices. Five edges meet at each vertex, but that counts each edge twice. Therefore, there are $\frac{1}{2}(5 \times 12) = 30$ edges. The total number of vertices and edges is $12 + 30 = 42$.

5. If X, Y and Z are consecutive positive integers with X > Y > Z, then $Y = Z+1$ and $X = Z+2$. Therefore, the value of XYZ is $100X + 10Y + Z = 100Z + 200 + 10Z + 10 + Z = 111Z + 210$. The sum of the digits of N is $Z + 2 + Z + 1 + Z = 3Z + 3$. Since the value of N is 48 times the sum of its digits we have $111Z + 210 = 144Z + 144 \Rightarrow 33Z = 66 \Rightarrow Z = 2 \Rightarrow N = 432$.

6. $Y = k \cdot \frac{ab}{c^2}$ so that $k \cdot \frac{(1.1a)(0.96b)}{(0.95c)^2} = k \cdot \frac{ab}{c^2} \cdot \frac{(1.1)(0.9)}{(0.95)^2} = k \cdot \frac{ab}{c^2} \cdot (1.170083\ldots) \Rightarrow Y$ would be increased by 17%.

7. Let $x = \sqrt{N + \sqrt{N + \sqrt{N + \ldots}}}$ so that $x = \sqrt{N + x} \Rightarrow x^2 = N + x \Rightarrow N = x^2 - x = x(x-1)$. Now, if $x(x-1) \le 2012$, the largest positive integer value of x for which this is true is $x = 45$, since $45 \cdot 44 = 1980$ (and $46 \cdot 45 = 2070$). As a result, $N = 1980$.

8. Using the diagram, $\tan^2 A + \tan^2 B = \left(\frac{a}{b}\right)^2 + \left(\frac{b}{a}\right)^2 = \frac{a^4 + b^4}{a^2b^2} = \frac{(a^2 - b^2)^2 + 2a^2b^2}{a^2b^2} = \frac{4a^2b^2 + 2a^2b^2}{a^2b^2} = 6$.

9. To get a zero at the end of the decimal expansion of the number, it takes a product of a 2 and a 5. There are more twos than fives in the product $(5^5)! = 3125!$, so if we count the number of fives, that will be equal to the number of zeros at the end of 3125! First, count one 5 in each of the numbers that has at least one 5 as a factor: 5, 10, 15, . . . , 3125 for a total of $\frac{3125}{5} = 625$ fives. Next, count an additional five in each number that has at least two fives as a factor: 25, 50, 75, ..., 3125 for a total of $\frac{3125}{25} = 125$ fives. Next, count an additional five in 125, 250, 375, ...,3125 for a total of $\frac{3125}{125} = 25$ fives. Then count an additional five in 625, 1225, 1875, ..., 3125 for a total of $\frac{3125}{625} = 5$ fives. Finally count an additional 5 in 3125 for one more. Therefore, the number of fives (and the number of zeros at the end of 3125!) is $625 + 125 + 25 + 5 + 1 = 781$.